METRIC SPACES: FINAL EXAM 2012

DOCENT: A. V. KISELEV

Evaluation: $\min\Bigl(100\%, \max\bigl(5\,\text{prb} \times 20\% \cdot \left[\begin{smallmatrix}1.00\\ 1.15^{\text{top}}\end{smallmatrix}\right],\ \sum_{i=1}^{6} \text{h/w} \times 5\% + 5\,\text{prb} \times 14\% \cdot \left[\begin{smallmatrix}1.00\\ 1.15^{\text{top}}\end{smallmatrix}\right]\bigr)\Bigr)$.

Problem 1. Let $(\mathcal{X}, d_{\mathcal{X}})$ be a non-empty complete metric space. Suppose that $f, g\colon \mathcal{X} \to \mathcal{X}$ are two contractions of $\mathcal{X}$. Does there always exist a point $x_0 \in \mathcal{X}$ such that $f\bigl(g(x_0)\bigr) = x_0$? (state and prove)

Problem 2 (top**)**.** Let $(\mathcal{X}, d_{\mathcal{X}})$ be a sequentially compact metric space. Consider a nested sequence $\mathcal{X} \supseteq S_1 \supseteq S_2 \supseteq \ldots \supseteq S_i \supseteq \ldots$ of non-empty closed subsets of $\mathcal{X}$, here $i \in \mathbb{N}$. Is it true that the intersection $\bigcup_{i=1}^{+\infty} S_i$ always remains non-empty? (state and prove)

- Does the answer remain the same or change if the hypothesis of the sequential compactness of $\mathcal{X}$ is discarded? Find a (counter)example supporting your claim.

Problem 3 (top)**.** Let A and B be connected subsets of a metric space and $A \cap \overline{B} \neq \varnothing$. Prove that the union $A \cup B$ is connected.

Problem 4. Are the two subsets of the Euclidean plane, $[0,1] \times [0,1)$ and $[0,1) \times [0,1)$, homeomorphic or not? (state and prove)

(By definition, a mapping $f\colon A \to B$ between two sets $A, B \subseteq \mathcal{X}$ of a metric space $(\mathcal{X}, d_{\mathcal{X}})$ is a homeomorphism if it is bijective and both f and f^{-1} are $d_{\mathcal{X}}$-continuous. The sets are homeomorphic if there is a homeomorphism between them.)

Problem 5. Let A and B be bounded subsets of a metric space $(\mathcal{X}, d_{\mathcal{X}})$ and $A \cap B \neq \varnothing$. Prove the inequality $\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B)$.

(By definition, $\operatorname{diam}(\varnothing) = 0$ and $\operatorname{diam}(S) = \sup_{x,y \in S} d_{\mathcal{X}}(x,y)$ for a non-empty bounded set $S \subseteq \mathcal{X}$.)

Date: April 4, 2012.

Do not postpone your success until July 4, 2012. GOOD LUCK!